

# The generalized evolution of linear bias: a tool to test gravity

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We derive an exact analytical solution for the redshift evolution of linear and scale-independent bias, by solving a second order differential equation based on linear perturbation theory. This bias evolution model is applicable to all different types of dark energy and modified gravity models. We propose that the combination of the current bias evolution model with data on the bias of extragalactic mass tracers could provide an efficient way to discriminate between “geometrical” dark energy models and dark energy models that adhere to general relativity.

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## 1. INTRODUCTION

It is well known that the large-scale clustering pattern of different extragalactic mass tracers (galaxies, clusters, etc) trace the underlying dark matter distribution in a biased manner [1] [2]. Such a biasing is assumed to be statistical in nature; with galaxies and clusters being identified as high peaks of an underlying, initially Gaussian, random density field. The linear and scale-independent bias factor,  $b$ , is thus defined as the ratio of the mass tracer overdensity to that of the underlying mass overdensity, or equivalently as the ratio of the square root of the mass tracer 2-point correlation function to that of the underlying mass correlation function. Furthermore, the redshift evolution of bias,  $b(z)$ , is very important in order to relate observations with models of structure formation and has been shown to be a monotonically increasing function of redshift.

There are two basic families of analytic bias evolution models. The first, called the *galaxy merging* bias model, utilizes the halo mass function and is based on the Press-Schechter [3] formalism, the peak-background split [2] and the spherical collapse model [4]. Many studies have compared the prediction of the *merging* bias model with numerical simulations and beyond an overall good agreement, differences have been found in the details of the halo bias. These differences have lead to modifications of the original model to include the effects of ellipsoidal collapse [5] and to either provide new fitting bias model parameters [6], or new forms of the bias model fitting function [7] or even a non-Markovian extension of the

excursion set theory [8].

The second family of bias evolution models assumes a continuous mass-tracer fluctuation field, proportional to that of the underlying mass, and the tracers act as “test particles”. In this context, the hydrodynamic equations of motion and linear perturbation theory are used. This family of models can be divided into two sub-families:

(a) The so-called *galaxy conserving* bias model uses the continuity equation and the assumption that tracers and underline mass share the same velocity field [9],[10],[11],[12]. Then the bias evolution is given as the solution of a 1st order differential equation, and Tegmark & Peebles [11] derived:  $b(z) = 1 + (b_0 - 1)/D(z)$ , with  $b_0$  is the bias factor at the present time and  $D(z)$  the growing mode of density perturbations. However, this bias model suffers from two fundamental problems: *the unbiased problem* ie., the fact that an unbiased set of tracers at the current epoch remains always unbiased in the past, and *the low redshift problem* ie., the fact that this model represents correctly the bias evolution only at relatively low redshifts  $z \lesssim 0.5$  [13]. Note that [14] has extended this model to also include an evolving mass tracer population in a  $\Lambda$ CDM cosmology.

(b) A model based on the basic equation for the evolution of linear density perturbations, and on the assumption of linear and scale-independent bias, which are used to derive a second order differential equation for the bias, the approximate solution of which provides the evolution of bias (see [15] and [16]). The provided solution applies to cosmological models, within the framework of general relativity, with a constant in time dark energy equation of state parameter (ie., quintessence or phantom).

In this article, we extend the original Basilakos & Plionis [15] bias evolution model to provide an exact solution valid for all dark energy and modified gravity cosmologies. This implies that the current bias evolution model

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can be used to put constraints on dark energy models as well as to investigate possible departures from general relativity.

## 2. THE EVOLUTION OF THE LINEAR GROWTH FACTOR

In this section, we discuss the basic equation which governs the behavior of the matter perturbations on sub-horizon scales and within the framework of any dark energy (hereafter DE) model, including those of modified gravity (“geometrical dark energy”). For these cases, a full analytical description can be introduced by considering an extended Poisson equation together with the Euler and continuity equations. Consequently, the evolution equation of the matter fluctuations, for models where the DE fluid has a vanishing anisotropic stress and the matter fluid is not coupled to other matter species (see [17],[18],[19],[20],[21],[22],[23]), is given by:

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\text{eff}}\rho_m\delta_m = 0 \quad (2.1)$$

where  $\rho_m$  is the matter density and  $G_{\text{eff}}(t) = G_N Y(t)$ , with  $G_N$  denoting Newton’s gravitational constant.

For those cosmological models which adhere to general relativity,  $[Y(t) = 1, G_{\text{eff}} = G_N]$ , the above equation reduces to the usual time evolution equation for the mass density contrast [24], while in the case of modified gravity models (see [17],[21], [22],[23]), we have  $G_{\text{eff}} \neq G_N$  (or  $Y(t) \neq 1$ ). In this context,  $\delta_m(t) \propto D(t)$ , where  $D(t)$  is the linear growing mode (usually scaled to unity at the present time). Changing variables from  $t$  to  $a$ , equation (2.1) becomes:

$$\frac{d^2\delta_m}{da^2} + A(a)\frac{d\delta_m}{da} - B(a)\delta_m = 0 \quad (2.2)$$

where

$$A(a) = \frac{d\ln E}{da} + \frac{3}{a} \quad \text{and} \quad B(a) = \frac{3\Omega_m}{2a^5 E^2(a)} Y(a) \quad (2.3)$$

with  $\Omega_m$  being the density parameter at the present time and  $E(a) = H(a)/H_0$  is the normalized Hubble function.

Useful expressions of the growth factor have been given by [24] for the  $\Lambda$ CDM cosmology. Several works have also derived the growth factor for  $w(z) = \text{const}$  DE models (see [25],[26],[27]), and for the braneworld cosmology [17]. Also Linder & Cahn [21] derived similar expressions for “geometrical” dark energy models in which the Ricci scalar varies with time, as well as for models with a time-varying equation of state, while for the scalar tensor and  $f(R)$  models the growth factors are provided by Gannouji et al. [23] and Tsujikawa et al. [22].

## 3. THE GENERAL EVOLUTION OF BIAS

In Basilakos & Plionis [15], we assumed that for the evolution of the linear bias, the effects of non-linear gravity and hydrodynamics (merging, feedback mechanisms

etc) can be ignored (see [10],[11]). Then, using linear perturbation theory in the context of general relativity  $[Y(t) = 1, G_{\text{eff}} = G_N]$  we obtained a second order differential equation which describes the evolution of the linear bias factor,  $b$ , between the background matter and the mass-tracer fluctuation field:

$$\ddot{y}\delta_m + 2(\dot{\delta}_m + H\delta_m)\dot{y} + 4\pi G_{\text{eff}}\rho_m\delta_m y = 0, \quad (3.1)$$

where  $y = b - 1$ . Below, we will prove that the above expression is valid for any cosmological model<sup>1</sup> including those of modified gravity, with  $G_{\text{eff}} = G_N Y(t)$ . Since we also make the same assumption, as in our original formulation, that the tracers and the underlying mass distribution share the same velocity field and thus the same gravity field, the above equation is valid also for cosmological models with a modified theory of gravity. Using the latter we have

$$\dot{\delta}_m + \nabla u \simeq 0 \quad \text{and} \quad \dot{\delta}_{\text{tr}} + \nabla u \simeq 0, \quad (3.2)$$

from which we obtain

$$\dot{\delta}_m - \dot{\delta}_{\text{tr}} = 0. \quad (3.3)$$

Now since we assume linear biasing, we have  $\delta_{\text{tr}} = b\delta_m$ , and using  $y = b - 1$ , we get that  $d(y\delta_m)/dt = 0$ . Differentiating the latter twice, we then get:  $\ddot{y}\delta_m + 2\dot{y}\dot{\delta}_m + y\ddot{\delta}_m = 0$ . Solving for  $y\ddot{\delta}_m$ , using the fact that  $y\dot{\delta}_m = -\dot{y}\delta_m$  and eq.(2.1) we finally obtain eq.(3.1).

Transforming equation (3.1) from  $t$  to  $a$ , we simply derive the evolution equation of the function  $y(a)$  [where  $y(a) = b(a) - 1$ ] which has some similarity with the form of eq.(2.2) as expected. Indeed this is

$$\frac{d^2 y}{da^2} + \left[ A(a) + \frac{2f(a)}{a} \right] \frac{dy}{da} + B(a)y = 0, \quad (3.4)$$

where  $f(a)$  is the growth rate of clustering, a parametrization of the linear matter perturbations, given by:

$$f(a) = \frac{d\ln\delta_m}{d\ln a} = \frac{d\ln D}{d\ln a} = \Omega_m^\gamma(a), \quad (3.5)$$

where  $\Omega_m(a) = \Omega_m a^{-3}/E^2(a)$  and  $\gamma$  is the growth index, originally introduced by Wang & Steinhardt [26]. Integrating eq.(3.5) we obtain the growth factor for any type of dark energy:

$$D(a) = a e^{\int_0^a (dx/x) [\Omega_m^\gamma(x) - 1]}. \quad (3.6)$$

In Basilakos & Plionis [15], we have provided an approximate solution of eq.(3.1), using  $f(z) \sim 1$  (which is

<sup>1</sup> The current theoretical approach does not treat the possibility of having interactions in the dark sector. Also discussions beyond the linear biasing regime can be found in [28] (and references therein).

valid at relatively large redshifts), but only in the framework of general relativity, i.e.,  $Y(t) = 1$ , which contains a quintessence (or phantom) dark energy. Here our aim is to provide a full analytical solution for all possible dark energy cosmologies that have appeared in the literature, such as a cosmological constant  $\Lambda$  (vacuum), time-varying  $w(t)$  cosmologies, quintessence,  $k$ -essence, quartessence, vector fields, phantom, modifications of gravity, Chaplygin gas etc.

Inserting now  $y(a) = g(a)/D(a)$  into eq.(3.4) and using simultaneously equation 2.2 and the second equality of equation 3.5, we obtain:

$$\frac{d^2g}{da^2} + A(a)\frac{dg}{da} = 0. \quad (3.7)$$

That is, the general solution of the latter equation is

$$g(a) = C_1 + C_2 \int \frac{da}{a^3 E(a)} \quad (3.8)$$

where  $C_1$  and  $C_2$  are the integration constants. Utilizing now  $a = (1+z)^{-1}$ ,  $b = y + 1 = (g/D) + 1$ ,  $b_0 = b(0)$  and eq.(3.8), we finally obtain the functional form which provides the evolution of linear bias for all possible types of DE models, including those of modified gravity, as:

$$b(z) = 1 + \frac{b_0 - 1}{D(z)} + C_2 \frac{J(z)}{D(z)} \quad (3.9)$$

where

$$J(z) = \int_0^z \frac{(1+x)dx}{E(x)}. \quad (3.10)$$

Since different halo masses result in different values of  $b_0$ , one should expect that the constants of integration  $C_1 = b_0 - 1$  and  $C_2$  should be functions of the mass of dark matter halos (see [16]), assuming that the extragalactic mass tracers are hosted by a dark matter halo of a given mass. Note that an extension of our model for the case of an evolving mass tracer population (i.e., including the effects of halo merging) is provided in appendix A.

Finally, comparing our solution of eq.(3.10) with that of the usual galaxy-conserving bias evolution model,  $b(z) = 1 + (b_0 - 1)/D(z)$ , it becomes evident that the latter misses one of the two components of the full solution. Furthermore, our full solution does not suffer from the *unbiased* and the *low redshift* problems, but more importantly, the dependence of our bias evolution model on the different cosmologies enters through the different behavior of  $D(a)$ , which is affected by  $\gamma$  (see equation 3.6), and of  $E(a) = H(a)/H_0$ .

It is interesting to mention that measuring the growth index could provide an efficient way to discriminate between modified gravity models and DE models which adhere to general relativity. Indeed it was theoretically shown that for DE models inside general relativity the growth index  $\gamma$  is well fitted by  $\gamma_{GR} \approx 6/11$  (see [21],[30]). Notice, that in the case of the braneworld model of

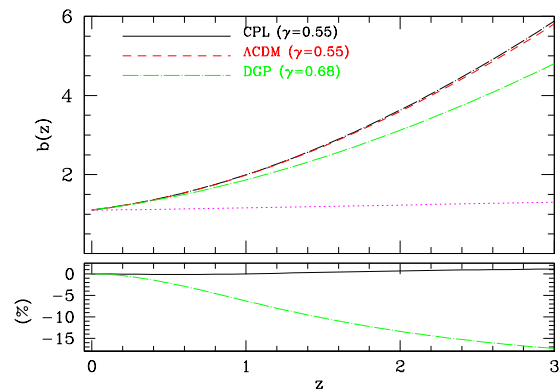


FIG. 1: The bias  $z$ -evolution for different flat cosmological models (*upper panel*) and their fractional difference with respect to the  $\Lambda$ CDM model (*lower panel*). The models shown are: CPL (solid line) with  $w(a) = w_0 + w_1(1 - a)$  and  $\gamma = 0.55$ , concordance  $\Lambda$ CDM (dashed line) and DGP (dot-dashed) with  $w(a) = -[1 + \Omega_m(a)]^{-1}$  and  $\gamma = 0.68$ . Note that we use  $\Omega_m = 0.27$ ,  $(w_0, w_1) = (-0.93, -0.38)$  [29],  $b_0 = 1.1$  and  $C_2 = 0.45$ . Finally, we also plot (dotted line) the bias evolution for  $C_2 = 0$  which corresponds to that of [11].

Dvali, Gabadadze & Porrati [31] (hereafter DGP) we have  $\gamma \approx 11/16$  (see also [21]). Indeed, it has been proposed (see [32]) that an efficient avenue to constrain the  $\gamma$  parameter is by determining observationally the redshift-dependent linear growth of perturbations. Alternatively other methods have been proposed in the literature, such as redshift space distortions in the galaxy power spectrum and the growth rate of massive galaxy clusters (see for example [33] and references therein). It is interesting to mention here that the above methods also assume a linear and scale-independent bias.

An alternative approach is to use the current generalized bias evolution, cosmology and  $\gamma$  dependent, relation and high quality observational bias data to test gravity. Of course, the observational bias data are derived for a particular cosmological model, but it is an easy task to scale them to each tested model in a consistent manner. Note, that such data are already available in the literature for the case of optical QSOs [34]. If the derived value of  $\gamma$  shows scale or time dependence or it is inconsistent with  $\gamma_{GR} \approx 6/11$ , then this will be a hint that the nature of dark energy reflects in the physics of gravity. Such an analysis is in progress and will be published elsewhere.

In order to visualize the redshift and  $\gamma$  dependence of our bias model, we compare in Fig. 1, a few flat cosmological models in which we impose  $\Omega_m = 0.27$ ,  $b_0 = 1.1$  and  $C_2 = 0.45$ . In particular we consider the following cases:

- (a) the CPL parametrization [35] with  $\gamma = 0.55$  (solid line),
- (b) the concordance  $\Lambda$ CDM ( $\gamma = 0.55$ , dashed line), and
- (c) the DGP with  $\gamma = 0.68$  (dot-dashed line).

The dotted line shows the bias evolution of Tegmark & Peebles [11] model, which is also described by our bias

model in the limit of  $C_2 = 0$ . In the lower panel of Fig.1 we show the fractional difference of the model bias with respect to that of the  $\Lambda$ CDM.

#### 4. CONCLUSIONS

In this work we provide a general bias evolution model, based on linear perturbation theory, which is valid for all possible non-interacting dark energy models, including those of modified gravity. Thus the current generalization of the bias evolution model can be viewed as a necessary step and an ideal tool to test the validity of general relativity on cosmological scales.

It is however important to spell out clearly which are the basic assumptions of our model, which are common also to many bias models in the literature: (a) the mass tracers and the underlying mass share the same velocity/gravity field, (b) the biasing is linear on the scales of interest (which does not preclude being scale dependent on small non-linear scales), and (c) that each dark matter halo is populated by one extragalactic mass tracer, which is an assumption that enters, at the present development of our model, only in the comparison of our model with observational bias data and not in the derivation of its functional form.

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#### Appendix A: BIAS EVOLUTION FOR AN EVOLVING MASS-TRACER POPULATION

Here we obtain the general linear bias evolution model assuming that the mass-tracer population evolves with time according to a  $(1+z)^\nu$  law. We now drop the assumption used in section 3, that the mass-tracer number density is conserved in time, by allowing a contribution from the corresponding interactions among the mass tracers. We obtain again the corresponding equation (3.1), starting from the continuity equation and introducing an additional time-dependent term,  $\Psi(t)$ , which we associate with the effects of interactions and merging of the mass tracers. We also make the same assumption, as in our original formulation, that the tracers and the underlying mass distribution share the same velocity field (or gravity field). Then:

$$\dot{\delta}_m + \nabla u \simeq 0 \quad \text{and} \quad \dot{\delta}_{\text{tr}} + \nabla u + \Psi(t) \simeq 0, \quad (\text{A.1})$$

from which we obtain:

$$\dot{\delta}_m - \dot{\delta}_{\text{tr}} = \Psi. \quad (\text{A.2})$$

Although we do not have a fundamental theory to model the time-dependent  $\Psi(t)$  function, it appears to depend

on the tracer number density and its logarithmic derivative as well as on the tracer overdensity:  $\Psi(t) \propto \Psi(\bar{n}, (1+\delta_{\text{tr}})d \ln \bar{n}/dt)$  (see eq.10 of [14] and appendix of Basilakos et al. [16]).

Now, in the context of linear biasing, we have  $\delta_{\text{tr}} = b\delta_m$  and utilizing  $b = y + 1$ , we find that  $d(y\delta_m)/dt = -\Psi$ . Differentiating twice the latter we then get:  $\ddot{y}\delta_m + 2\dot{y}\dot{\delta}_m + y\ddot{\delta}_m = -\dot{\Psi}$ . Solving for  $y\ddot{\delta}_m$ , using the fact that  $y\dot{\delta}_m = -\dot{y}\delta_m - \Psi$  and equation (2.1) we arrive at the following expression:

$$\ddot{y}\delta_m + 2(\dot{\delta}_m + H\delta_m)\dot{y} + 4\pi G_{\text{eff}}\rho_m\delta_m y = -2H\Psi - \dot{\Psi} \quad (\text{A.3})$$

which is the corresponding equation (3.1) for the case of interactions among the tracers.

Transforming again equation (2.1) from  $t$  to  $a$ , we get

$$\frac{d^2 y}{da^2} + \left[ A(a) + \frac{2f(a)}{a} \right] \frac{dy}{da} + B(a)y = F(a) \quad (\text{A.4})$$

where

$$F(a) = -\frac{2\Psi(a) + a(d\Psi/da)}{a^2 D(a)H(a)}. \quad (\text{A.5})$$

Now, following the same notations ( $y = g/D$ ) as in section 3 the above differential equation becomes:

$$\frac{d^2 g}{da^2} + A(a)\frac{dg}{da} = F(a). \quad (\text{A.6})$$

Integrating eq.(A.6) it is straightforward to estimate the general solution of the bias factor. This is

$$g(a) = C_1 + C_2 \int \frac{da}{a^3 E(a)} + \int \frac{da}{a^3 E(a)} \int^\alpha F(\tilde{a}) \tilde{a}^3 E(\tilde{a}) d\tilde{a}. \quad (\text{A.7})$$

Using the same conditions with those provided in section 3, the bias evolution in the redshift space takes the form

$$b(z) = 1 + \frac{b_0 - 1}{D(z)} + C_2 \frac{J(z)}{D(z)} + \frac{y_p(z)}{D(z)} \quad (\text{A.8})$$

where

$$y_p(z) = \int_0^z \frac{(1+x)}{E(x)} dx \int_0^x \frac{F(u)E(u)}{(1+u)^5} du. \quad (\text{A.9})$$

Obviously, if the interaction among the tracers is negligible ( $\Psi \simeq 0$ ) then eq.(A.8) boils down to eq.(3.10) as it should.

Now, knowledge of the exact functional form of the interaction term  $\Psi(z)$  would provide the precise redshift evolution of the bias. As we have analytically proved in the appendix of Basilakos et al. [16], a reasonable approach regarding the evolutionary  $\Psi(z)$  term is that:  $\Psi(z) = AH_0(1+z)^\nu$ , where  $\nu \sim 3$ . Note that the Hubble constant has been maintained for mathematical convenience. Inserting the latter equation,  $a = (1+z)^{-1}$  and  $d\Psi/da = -(1+z)^2 d\Psi/dz$  into the second term of eq.(A.4) we derive that

$$F(z) = A(\nu - 2) \frac{(1+z)^{\nu+2}}{D(z)E(z)}, \quad (\text{A.10})$$

where  $A$  is a positive parameter (to be determined from observational data see Basilakos et al. *in preparation*). Obviously, for  $\nu > 2$  the derived bias evolution becomes stronger than in the case of no interactions, especially at high redshifts, which means that due to the merging processes the halos (of some particular mass) correspond to higher peaks of the underlying density field with respect to equal mass halos in the non-interacting case. On the

other hand, the  $\nu < 2$  case corresponds to the destruction of halos of a particular mass, which results into a lower-rate of bias evolution with respect to the non-interacting case. Now, for the limiting case with  $\nu = 2$  we obtain  $y_p = 0$ , implying no contribution of the interacting term to the bias evolution solution, as in the case with  $\Psi = 0$ , which can be interpreted as the case where the destruction and creation processes are counter-balanced.

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